Renormalization group

Three phases: Removal of UV divergences (local, coupling-by-coupling)
Critical phenomena (local, coupling-by-coupling)
Algorithm to solve strongly coupled systems (global, functional)

1. Nonperturbative method (beyond resummation)
2. Anderson localisation
3. Global RG
4. Condensation
5. Real time dynamics, quantum-classical crossover
Nonperturbative method

Q: How to calculate integrals nonperturbatively?
A: No explicit calculation, evolution only.

Functional differential equation:

\[ I_g(k) = \int dx e^{iS_g(x) + ikx} \]

\[ \partial_g I_g(k) = \int dx e^{iS_g(x) + ikx} \partial_g S_g(x) \]

\[ = \partial_g S_g \left( -i \frac{\partial}{\partial k} \right) I_g(k) \]

Solved by numerical integration in \( g \):

\[ I_g(k) = I_{g_0}(k) + \int_{g_0}^g dg' \partial_{g'} S_{g'} \left( -i \frac{\partial}{\partial k} \right) I_{g'}(k) \]

New small parameter: \( \frac{\delta g}{g}, \quad g = \Lambda, m, \hbar, e, \ldots \)
Realization: $I_{g}(x) \rightarrow$ effective action

\[ e^{iW_{g}[j]} = \int D[\phi] e^{iS[\phi]+\frac{1}{2} \int dx dy \phi(x) K_{g}(x-y) \phi(y)} + i \int dx j(x) \phi(x) \]

\[ \Gamma[\phi] = W[j] - \int dx j(x) \phi(x), \quad \phi(x) = \frac{\delta W[j]}{\delta j(x)} \]

\[ \Gamma[\phi] = \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_{1} \cdots dx_{n} \Gamma^{(n)}(x_{1}, \ldots, x_{n}) \phi(x_{1}) \cdots \phi(x_{n}) \]

**Evolution (RG) equation:** Initial condition $\Gamma_{g_{0}}[\phi] = S[\phi]$

\[ \partial_{g} W_{g}[j] = -e^{-iW_{g}[j]} \int dx dy \frac{\delta}{\delta j(x)} \partial_{g} K_{g}(x-y) \frac{\delta}{\delta j(y)} e^{iW_{g}[j]} \]

\[ \partial_{g} \Gamma_{g}[\phi] = \frac{1}{2} \text{Tr} \left[ \partial_{g} K \frac{1}{\delta \phi \delta \phi} + K_{g} \right] + \frac{1}{2} \int dx dy \phi(x) \partial_{g} K_{g}(x-y) \phi(y) \]

**Conventional RG:** few coupling constants, $\infty$ orders in pert. exp.

**Functional, differential RG:** $\infty$ many coupling constants, first order in pert. exp.

better truncation schemes
Anderson localisation
(Free electrons, 3D on $160^3 - 320^3$ lattices, $\hbar = a = m = 1$)

Inverse conductivity
strength of disorder
$0.017 < \rho < 0.5$

Inverse conductivity
frequency
$\rho = 0.11$,
$g = 10, 10^2, 10^3, 10^4, 10^5$

$\sqrt{\frac{g \rho x}{E_{\text{band}}}}$
density
$\omega = 0.00512$
and
$\omega = 0.00002$

Inclusion of (strong) Coulomb interaction is possible
High energy physics - Critical phenomena connection

Asymptotic scaling laws around a fixed point: relevant (↗) and irrelevant (↘) operators

Critical phenomena: $a ↗$
High energy physics $a = \frac{2\pi}{\Lambda} ↘$

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<tr>
<th>Statistical Physics</th>
<th>Quantum Field Theory ($\hbar = c = 1$)</th>
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<tr>
<td>lattice spacing, minimal distance: $a$</td>
<td>cutoff: $\Lambda = \frac{2\pi}{a}$</td>
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<tr>
<td>correlation length: $\xi$</td>
<td>Compton wavelength: $\frac{1}{m}$</td>
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<tr>
<td>critical phenomenon: $a$ is fixed, $\frac{\xi}{a} \rightarrow \infty$</td>
<td>renormalization: $m$ is fixed, $\frac{\Lambda}{m} = \frac{2\pi}{\xi} = 2\pi \frac{\xi}{a} \rightarrow \infty$</td>
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<tr>
<td>UV fixed point</td>
<td>renormalized theory</td>
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<td>relevant or marginal operator</td>
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<td>irrelevant operator</td>
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<td>universality</td>
<td>renormalizable theories cover all possible dynamics</td>
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</table>
Global RG
(J. Alexandre, V. Branchina, J. Polonyi)

Solids:
UV scaling law: $\gamma, e, p$
IR scaling law: $\gamma, e_{\text{cond}}, $ phonons

Four possibility for global assignment:

$g$

$q_e$

$(\bar{\psi}\psi)^2$

$q_{\mu}$

$(\bar{\psi}\psi)^3$
Global RG for the Theory Of Everything

Passing by fixed points
Islands of local universality
Bifurcations: Phase transitions
Chaotic trajectory?
Fundamental vs. applied physics
**RG microscope**
J. Alexandre, V. Branchina, S. Nagy, I. Nandori, J. Polonyi, K. Sailer

**B-E condensate**  (eg. supercurrent density in a solid as a function of the parameters beyond the Standard Model)

**Quark confinement**  Haar measure of the gauge group

Massive sine-Gordon model:  $k$ gliding cut-off

$$L = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + u \cos(\beta \phi) + \frac{M^2}{2} \phi^2$$

Sensitivity: $\frac{\partial u(k)}{\partial u(\Lambda)}$
Microscope effect in the molecular phase

$\beta^2 < 8\pi$
Condensation
V. Pangon, S. Nagy, J. Polonyi, K. Sailer

**Wegner-Houghton equation:** lowering the cutoff \( k \to k - \Delta k \)

\[
e^{-\frac{i}{\hbar}S_{k-\Delta k}[\phi]} = \int D[\tilde{\phi}] e^{-\frac{i}{\hbar}S_k[\phi+\tilde{\phi}]}
\]

\[
S_k[\phi + \tilde{\phi}_c] - S_{k-\Delta k}[\phi] = -\frac{\hbar}{2} \text{Tr} \ln \frac{\delta^2 S_k[\phi + \tilde{\phi}_c]}{\delta \phi \delta \phi} + \mathcal{O}(\hbar^2)
\]

Ansatz: \( S_k[\phi] = \int d^4x \left[ \frac{1}{2} (\partial \phi)^2 + V_k(\phi) \right] \), \( \tilde{\phi}_c = 0 \)

\[
\partial_k V_k(\phi) = -\frac{\hbar k^3}{16\pi^2} \ln[k^2 + \partial_c^2 V_k(\phi)]
\]
\[ \partial_k V_k(\phi) = -\frac{\hbar k^3}{16\pi^2} \ln[k^2 + \partial^2_\phi V_k(\phi)] \]

force the restoration of equilibrium
Condensation $\leftrightarrow$ SSB $\leftrightarrow \partial^2_\phi V(0) < 0$

\[ k^2 + \partial^2_\phi V_k(\phi) = 0: \text{no analytical approaches} \]
\[ k^2 + \partial^2_\phi V_k(\phi) < 0: \hat{\phi}_c \neq 0 \]
saddle point exp.

dynamical Maxwell-cut (deg. action)
(Branchina, Alexandre, Polonyi)
Approach of degeneracy

\[ V_B(\phi) = g \cos \beta \phi \]

\[ V_B(\phi) = \frac{1}{2} m_B^2 \phi^2 + \frac{1}{4} g_B \phi^4 \]

Quantum censorship?
Real time dynamics (CTP, J. Schwinger)

Variational formalism with initial conditions and friction forces

No variation at the end points

whatever order of the differential eq.

because \( \delta S[x] = p \neq 0 \)

CTP: Still \( \delta(x_{t_f}) = 0 \) \( (t_f \text{ arbitrary}) \)

but \( x(t) = \begin{cases} x^+(t) & 0 < t < t_f \\ x^-(t_f - t) & t_f < t < 2t_f \end{cases} \)

- Reduplication of the degrees of freedom \( \implies \) friction and retarded interactions

- Quantum case: Heisenberg representation, \( A(t) = e^{itH}A_se^{-itH} \)

Expectation values rather than transition amplitudes
Quantum-classical crossover

Density matrix:

\[ \langle x^{(+)}|\rho(t)|x^{(-)}\rangle = \langle x^{(+)}|e^{-i t H} \rho_0 e^{i t H} |x^{(-)}\rangle = \int D[x^\pm(t)] e^{i \frac{\hbar}{2} (S[x^+]-S[x^-])} \]

Decoherence: a necessary condition for classical physics

\[ \langle x+y|\rho(t)|x-y\rangle \text{ decreases for when } y \text{ increased} \]

Contribution with \( x^+(t) \not= x^-(t) \) suppressed in the path integral

Strong coupling regime: \( \langle x^{(+)}|\rho(t)|x^{(-)}\rangle \approx \int D[x] e^{i \frac{\hbar}{2} (S[x]-S[x])} = \int D[x] \)

Feynman’s scenario with rigid trajectories is invalid

Need of nonperturbative scheme \( \Rightarrow \) Quantum RG